Parameterized complexity for network optimization problems

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Lamsade Spring School

Parameterized complexity

After mastering computational complexity, the obvious realization is that most problems of interest are difficult :(

How to solve them optimally but also somehow efficiently?

Parameterized complexity $\rightarrow$ use additional information on the problem (=parameter) to:

- identify the 'causes' of the complexity
- (maybe) confine the unavoidable combinatorial explosion to some specific parameters

Parameterized problem

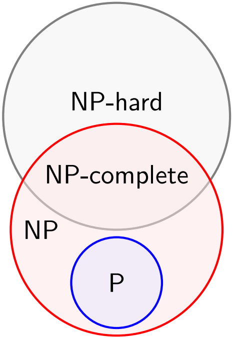
An instance of a **parameterized problem** is (x, k) with x the usual instance and k the parameter.
The parameter is some info on

- the output data ($\textsc{Clique}$ parameterized by the size of a solution), or
- the input data (SAT parameterized by the number of variables).

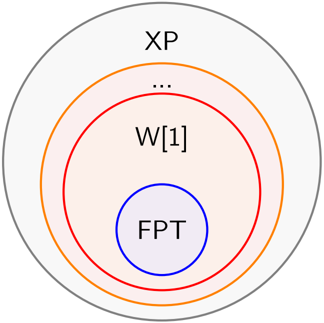
A **parameterized reduction** from (x, k) to (x', k') verifies:

- (x, k) is a yes-instance iff (x', k') is a yes-instance
- $k' \leq g(k)$ for some computable function g
- runtime bounded by $f(k) \cdot |x|^{O(1)}$ for some computable function f

Parameterized hierarchy

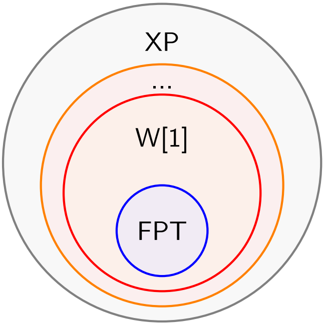


Classical complexity



Parameterized complexity

Parameterized hierarchy



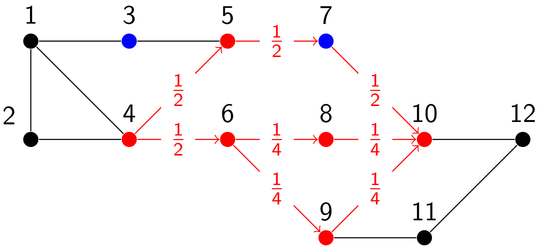
XP class: *slice-wise polynomial* problems
$\rightarrow f(k) \cdot |(x,k)|^{g(k)}$

FPT class: *fixed-parameter tractable* problems
$\rightarrow f(k) \cdot |(x,k)|^{c}$

Conjecture: $FPT \neq W[1]$

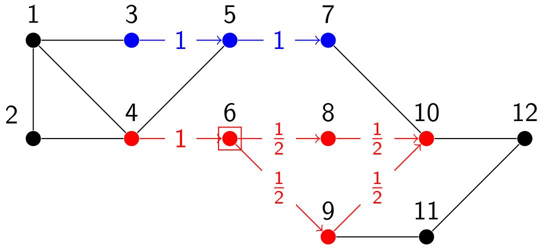
Network optimization problem

IP network: shortest paths and Equal Cost Multipath (ECMP)



Segment routing

A segment path using 6 as waypoint:



Segment Routing

Segment Routing Problem (SRP)

Input:

- undirected graph G, capacity function $c : E(G) \to \mathbb{N}$, weight function $\omega : E(G) \to \mathbb{N}$
- set of demands $D = \{(s_i, t_i, b_i) : i = 1, ..., d\}$
- budget $k \in \mathbb{N}$.

Is there a feasible routing scheme for D in G using at most k waypoints for each demand?

Unit Segment Routing: restricted version where $c(e) = 1$, $\omega(e) = 1$ and $b_i = 1$.

Parameterization

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Study of the problem

Basic brut force gives something like $O(|V(G)|^{kd}|E(G)|kd)$
$\rightarrow \mathrm{SRP} \in XP$ for parameter (k, d)

param	G undirected	G directed
d	NP-hard for $d = 4$	NP-hard for $d = 3$
k	probably NP-hard for $k = 1$	$\leftarrow$
$k + d$	probably W[1]-hard for $k = 1$	$\leftarrow$
vc τ	NP-hard for $\tau = 4$	
$\lvert V \rvert$ and $k = 1$	W[1]-hard	$\leftarrow$

Maybe polynomial on cactus graphs and wheels.

Conclusion

Thanks to parameterized complexity, we know that this problem is way too difficult except maybe on some very specific classes of graph for which a dedicated algorithm must be designed.